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2004 J. Phys. A: Math. Gen. 37 10007

(http://iopscience.iop.org/0305-4470/37/43/001)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 10007–10035 PII: S0305-4470(04)76919-X

Factorization: little or great algorithm?

Bogdan Mielnik and Oscar Rosas-Ortiz

Departamento de Física, CINVESTAV, AP 14-740, México 07000, DF, Mexico

E-mail: bogdan@fis.cinvestav.mx and orosas@fis.cinvestav.mx

Received 1 March 2004, in final form 9 June 2004
Published 14 October 2004
Online at stacks.iop.org/JPhysA/37/10007
doi:10.1088/0305-4470/37/43/001

Abstract
The progress of the factorization method since the 1935 work of Dirac is briefly reviewed. Though linked with older mathematical theories the factorization seems an autonomous 'driving force', offering substantial support to the present day Darboux and Bäcklund approaches.

PACS numbers: 03.65.Ca, 03.65.Ge, 03.65.Fd

1. Introduction

Some exceptional properties of our universe [1, 2] make possible the existence of atoms, galaxies and humans. Some doctrines say that our universe is not unique. New 'baby universes' are constantly born [3, 4]: some of them unable to host stars and galaxies, some *misanthropic* (no humans), some collapsing fast—and of course, we live in that which permits our existence! In a round table discussion (Warsaw 1988) a provocative question was asked: how it is that physical bodies typically interact with oscillator and Coulomb potentials—those for which the motion equations can be exactly solved? So, was the universe created specially to make possible our science? A voice from the public objected: since we are constructed (*grosso modo*) of Coulomb and oscillator potentials, our minds created a math in which these potentials are solvable. In other universes, in which the physical laws could be altered, intelligent beings would develop a distinct type of mathematics in which a different class of potentials would be exactly solvable. Can we guess this kind of mathematics? Unfortunately, chances are low. Yet, we can develop some techniques which already a century ago permitted new classes of exactly solvable spectral problems to be obtained. One of them exerts a special influence on our way of thinking.

Below, we report the progress of the factorization method in its most elementary form; we shall show that each step of the method is translated into some new physical ideas.

Since our review crosses several areas where distinct notation and units are used, we try to respect all particular traditions (without explaining every time), i.e., when discussing the physical results, our Schrödinger Hamiltonians will have the traditional form $H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x)$ and the 'creation' and 'annihilation' operators will be $A_\pm = \frac{1}{\sqrt{2}}\left(\mp \frac{d}{dx} + x\right)$ (the physicist would like to see the particle number N accepting the eigenvalues $n = 0, 1, 2, \ldots$). However, if referring to mathematical works, we shall use the 'Hamiltonians' $H = -\frac{d^2}{dx^2} + u(x)$ (the mathematicians would be outraged by the unnecessarily complicated 1/2, affecting the simplicity of their formulae as well as the traditional KdV coefficients).

2. The method of factorization

The credit for the factorization is usually given to Schrödinger [5, 6], but the technique appears first in Dirac's book [7] as a little stratagem to solve the spectral problem for the one-dimensional quantum oscillator. The idea was that the oscillator Hamiltonian can be written in terms of two first-order differential operators:

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 = A^\dagger A + \frac{1}{2}, \tag{2.1}$$

$$A = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right) = \frac{1}{\sqrt{2}}\, e^{-\frac{x^2}{2}} \frac{d}{dx}\, e^{\frac{x^2}{2}} \tag{2.2}$$

with the corresponding formula for $A^\dagger$, and

$$[A, A^\dagger] = 1, \qquad HA^\dagger = A^\dagger(H+1), \qquad HA = A(H-1). \tag{2.3}$$

Expressions (2.1)–(2.3) show immediately the existence of the ground state $|0\rangle$ and allow us to generate explicitly the higher energy states, without integrating any differential equation:

$$A|0\rangle = 0 \quad \Rightarrow \quad |0\rangle = \frac{1}{\sqrt{2}}\, e^{-\frac{x^2}{2}} \tag{2.4}$$

$$|n\rangle = \frac{1}{\sqrt{n!}}(A^\dagger)^n|0\rangle = c_n H_n(x)\, e^{-\frac{x^2}{2}}; \qquad H_n(x) = (-)^n\, e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\, e^{-\frac{x^2}{2}}. \tag{2.5}$$

Later, some authors considered the Dirac 'stratagem' as just an accidental trick, too limited to replace the genuine Sturm–Liouville problem. Yet, very soon, the 'little trick' dominated almost all quantum physics. Indeed, we became dependent: instead of being our *tools*, A, $A^\dagger$ turned *our way of thinking*. We stick to them even in adverse situations, e.g. looking for QFT in non-inertial frames (no physical sense granted [8]); or in curved space–time, where the vacuum $|0\rangle$ and the operators $A_k^\dagger$, A_l are not uniquely defined. (So, to which of many vacua should the system jump if induced to radiate? We do not know, yet we apply A, $A^\dagger$ [9]).

Soon, it was proved that the algorithm is not at all limited to the harmonic oscillators. The works of Schrödinger, Infeld *et al* [5, 6, 10–12] identified four classes of Hamiltonians admitting an exact factorization treatment. Each one is a finite (or discrete) family of structurally similar Hamiltonians H_m, *intertwined* by a sequence of 'creation' and 'annihilation' operators $A_m^\dagger$, A_m:

$$\begin{aligned} &A_m^\dagger A_m = H_{m-1} + \varepsilon_m, \qquad && A_m A_m^\dagger = H_m + \varepsilon_m; \quad m = 1, 2, \ldots \\ &\Rightarrow A_m H_{m-1} = H_m A_m; \qquad && H_{m-1}A_m^\dagger = A_m^\dagger H_m. \end{aligned} \tag{2.6}$$

The chain breaks if one arrives at the vector 0, and this is precisely the condition defining the discrete spectrum (cf [13]). The method shortens remarkably the solutions of the known eigenproblems (thus, e.g., one can construct immediately the *spherical functions*, the 'little Bessels', hypergeometric functions [14, 15], etc, without pilgrimage to the handbooks of special functions; see the remarks by Infeld [10]).

The further development shows an additional flexibility of the scheme which permits one to go beyond the four Infeld–Hull classes [12] by constructing the 'deformed factorizations'. As it seems, this aspect first appears in a paper of Deift [16] (conceived independently of the physical trend [5–7, 10–12]). Returning to the original Sturm *et al* works [17–19], Deift considers a pair of operators $A = b\frac{\mathrm{d}}{\mathrm{d}x}b^{-1}$, $A^\dagger = -b^{-1}\frac{\mathrm{d}}{\mathrm{d}x}b$, where b is differentiable, without nodal points in $\mathbb{R}$. Then

$$A^\dagger A \equiv H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \qquad AA^\dagger \equiv \widetilde{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \widetilde{V}(x) \tag{2.7}$$

where $V(x) = b^{-1}b''$ and $\widetilde{V}(x) = b(1/b)''$, so b and $\widetilde{b} = b^{-1}$ fulfil the eigenequations

$$b'' - V(x)b = 0; \qquad \widetilde{b}'' - \widetilde{V}(x)\widetilde{b} = 0 \tag{2.8}$$

with the potentials interrelated by

$$\widetilde{V}(x) = V(x) - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln b(x). \tag{2.9}$$

For simplicity, we skip the problem of norms and Hilbert space domains; they will be separately addressed if necessary. If b has no nodal points, $\widetilde{H}$ has no new singularities and $\widetilde{b}$ is an eigenfunction of $\widetilde{H}$ (typically, $\widetilde{b}$ is normalizable if b is not). Some more assumptions about the structure of H are adopted in [16], e.g. that it has a finite number of bound states which can be either deleted or added one by one. This becomes useful in an elegant solution of the inverse spectral problem [20], but the general consequences of the algorithm go beyond that. Thus, e.g., the deformed factorization applied to the harmonic oscillator (via the Riccati equation [21]) shows that the well-known Abraham–Moses potentials [22, 23] are a natural product of the commutation method [24]. When applied properly, the algorithm leads also to a new class of hydrogen-like potentials on $[0, +\infty)$ without new singularities [25].

The 'strategm' has soon opened some new windows into the future and into the past.

3. The Darboux heritage

For several decades the methods of Bäcklund and Darboux were applied in the mathematical theory of solitons [26, 27]. However, the true revival occurred afterwards. In 1984 Andrianov *et al* [28–31] showed that the use of the ground state eigenfunction to transform the Schrödinger operator (see, e.g., [16]) was not an accident but it had deeper roots in the 19th century Darboux result [32]. The fragment of Darboux's work which attracted so much attention was the simple statement: if $u(x)$ fulfils the second-order differential equation $-u'' + [V(x) - \varepsilon]u = 0$ and if $-\theta'' + V(x)\theta = 0$, then the function

$$\widetilde{u} = \left(-\frac{\mathrm{d}}{\mathrm{d}x} + \theta'/\theta\right)u \tag{3.1}$$

solves the new second-order equation $-\widetilde{u}'' + (\widetilde{V}(x) - \varepsilon) = 0$ with

$$\widetilde{V}(x) = V(x) + \theta\frac{\mathrm{d}^2}{\mathrm{d}x^2}\theta^{-1} \tag{3.2}$$

(see [32] p 1458). Expression (3.1) can be easily recognized as the ladder operator of the factorization method. It was henceforth concluded that all the 20th century quantum mechanical strategms are descendants of the Darboux theorem. The same point was raised by Luban and Pursey [33–35], who suggested that all the results derived from the factorization method (since Dirac and Schrödinger [5–7]) are nothing but applications of the Darboux method. This last statement, though, must be taken 'with a grain of salt'[1]. Looking carefully, the Darboux theorem was indeed a common denominator of many new results. But not the only one! What gave the entire 20th century trend its exceptional vitality was indeed the commutation algorithms derived from the 'little trick' (2.1). Darboux ignored this particular aspect; he could not predict that his theorem would be obtained some day just by commuting the operators. Yet, it was precisely the simplicity of the factorization which rescued his theorem from the past. Moreover, even if superficial, the 'commutation algorithm' almost has some of the power of medieval spells: a few simple words just calling to existence some new forces of nature . . . .

While the Darboux heritage is rescued, it is worth remembering this particular circumstance.

4. Is our universe supersymmetric?

An additional message of the commutation rules was for some time hidden under the calculational noise of quantum field theories (QFT). Yet, already in 1965–1970 a new type of symmetry coupling the bosonic (tensorial) and fermionic (spinorial) degrees of freedom was being studied by Miyazawa [36] and by Golfand and Lightman [37]. The next important step (taken independently) was due to Volkov and Akulov [38, 39] and the subsequent breakthrough (independent as well) appears in Wess and Zumino [40] (see also Zumino [41]), where the fermions and bosons have an equal status in the fermion–boson multiplets due to the *supergauge* transformations [40, 42–44]. The main attractive force of the new symmetry was the cancellation of the vacuum energies of the fermionic and bosonic components raising the hopes that one might avoid at least a part of infinities of non-renormalizable QFT (e.g. in quantum gravity.) However, a practical difficulty was the 'dead wood' of QFT (an ironic title of Dirac's paper [45].) Finally, in his 1981 work [46], Witten identifies the 'elementary cell' of the theory, later known as *supersymmetric quantum mechanics* (SUSY QM); see also [47]. Quite curiously, its mathematical skeleton coincides with the 'little stratagem' of Dirac *et al* [5–7, 10–12, 16].

In its simplest form, it involves just a pair of bosonic and fermionic 'nests' which can be occupied by n_B bosons and n_F fermions ($n_B = 0, 1, 2, \ldots; n_F = 0, 1$), with the Hilbert space of states $\mathcal{H}$ in the form of the tensor product $\mathcal{H}_B \otimes \mathcal{H}_F$ spanned by the Fock basis $|n_B, n_F\rangle = |n_B\rangle|n_F\rangle$. If the bosons and fermions do not interact, the system admits a simple representation as the bosonic$\otimes$fermionic oscillator, with the Fock states generated by the corresponding creation and annihilation operators $\sigma_\pm$, $A_\pm$ (please identify A_-, A_+ with A, $A^\dagger$ of section 2):

$$|n_B, n_F\rangle = \frac{1}{\sqrt{n_b!}} A_+^{n_B} \sigma_+^{n_F} |0_B, 0_F\rangle \tag{4.1}$$

where $\sigma_\pm^2 = 0, [A_-, A_+] = 1 = \{\sigma_-, \sigma_+\}$, and $[\cdot,\cdot]$, $\{\cdot,\cdot\}$ mean the commutator and anticommutator, respectively. In order not to complicate notation we shall use the same symbols $A_\pm$ to denote the bosonic operators acting in $\mathcal{H}_B$, as well as in $\mathcal{H}_B \otimes \mathcal{H}_F$ (in this

[1] Indeed, the entire development shows chronological gaps and inconsistencies; the ideas emerge, disappear and re-emerge again. In the idealized story, Darboux discovered his 1882 theorem which was then applied by Dirac *et al*, generalized by Crum and Krein. But in the real history, Dirac *et al* knew nothing about the Darboux theorem; the real Crum ignored that he was generalizing the Darboux method; he apparently did not care about Schrödinger [5, 6] and others; Krein did not know about Darboux, Deift refers only to Crum but neglects the entire physical trend, some other papers [24] follow Infeld and Hull, but ignore Deift, and so on.

last case they affect just the bosonic parts of the state vectors (4.1); the opposite for $\sigma_\pm$.) The Hamiltonian reads

$$\mathsf{H} = \left(A_+A_- + \tfrac{1}{2}\right)\omega_B + \left(\sigma_+\sigma_- - \tfrac{1}{2}\right)\omega_F, \tag{4.2}$$

where ω_B and ω_F are the one boson and one fermion energies. If now $\omega_B = \omega_F = \omega > 0$, the vacuum contribution to the energy cancels:

$$\mathsf{H} = \omega A_+A_- + \omega\sigma_+\sigma_-. \tag{4.3}$$

Vacuum $|0\rangle = |0_B, 0_F\rangle$ is still unique, but the higher energy levels of H are degenerate, spanned by pairs of eigenvectors which differ only by replacing one boson by one fermion or vice versa [48]. The fact is naturally expressed by introducing the 'permuting operators' $Q_- = A_+\sigma_-$, $Q_+ = A_-\sigma_+$ (called the *supercharges*) and by noting that H commutes with $Q_\pm$. The structure of H is then conveniently represented by using two orthogonal projectors $P_1 = \sigma_+\sigma_- = |1_F\rangle\langle 1_F|$ and $P_0 = \sigma_-\sigma_+ = |0_F\rangle\langle 0_F|$ onto subspaces with or without one fermion. By adopting the Pauli matrix representation of the fermionic operators, $\sigma_1 = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $\sigma_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$, $\sigma_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2) = \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$, $P_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, $P_0 = \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$, one obtains

$$\mathsf{H} = \omega A_+A_-P_0 + \omega A_-A_+P_1 = \begin{pmatrix} H_1 & 0\\ 0 & H_0\end{pmatrix} \tag{4.4}$$

where $H_0 = \omega A_+A_-$ and $H_1 = \omega A_-A_+$ are the bosonic and fermionic Hamiltonians (more exactly, both are *bosonic*: H_0 in the absence, H_1 in the presence of the one fermion permitted). In the standard convention ($\omega = 1$),

$$Q_- = \begin{pmatrix} 0 & 0\\ A_+ & 0\end{pmatrix}, \qquad Q_+ = \begin{pmatrix} 0 & A_-\\ 0 & 0\end{pmatrix} \tag{4.5}$$

with $Q_\pm$ and H obeying the supersymmetric (SUSY) algebra

$$\mathsf{H} = \{Q_-, Q_+\}, \qquad [\mathsf{H}, Q_\pm] = 0, \qquad Q_\pm^2 = 0. \tag{4.6}$$

The SUSY rules (4.6) assure the well-known spectral degeneracy of H [48]. However, they neither require the traditional oscillator form of $A_\pm$ nor the commutation rule $[A_-, A_+] = 1$. To satisfy (4.6), it is sufficient to employ an operator pair $A_\mp = A_\pm^\dagger = \frac{1}{\sqrt{2}}(\pm\mathrm{i}p + \alpha(x))$, where x and p are two real operators in $\mathcal{H}_B$ fulfilling $[x, p] = \mathrm{i}$. Then by adopting the representation of the state vectors $\psi \in \mathcal{H}_B$ as the 'wavefunctions' $\psi = \{\psi(x)\}$, with $p = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}$, one arrives at the bosonic and fermionic parts $H_0 = A_+A_-$ and $H_1 = A_-A_+$ in the form of two Schrödinger Hamiltonians

$$H_i = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha^2(x) + (-1)^{\mathrm{i}}\alpha'(x) \tag{4.7}$$

intertwined by the mechanism of [5, 6, 10–12, 16, 24, 25, 49]. As it seems, the equivalence of both designs was almost simultaneously noted by Andrianov *et al* [28] and by Nieto [50]. If, furthermore, $\alpha(x)$ has an adequate boundary behaviour at $x \to \pm\infty$, then the Hamiltonian H_0 has the ground state $|0_B\rangle$ with the eigenvalue $E_0 = 0$, absent in the spectrum of H_1; all other eigenvalues $E_1, E_2, \ldots,$ of H_0 coincide with the spectrum of H_1, granting again the double degeneracy of the higher levels of H. The spectra of H_0 and H_1, in general, do not need to be equally spaced; they paint a picture of a self-interacting bosonic field occupying two anharmonic oscillator ladders in two supersymmetric sectors of $\mathcal{H}$ ([48, 49], see also Stedman [51], section 3) where each bosonic state is coupled by $Q_\pm$ with its partner in one fermion sector (e.g. graviton with gravitino, etc). By observing the evolution of the subject, one cannot overlook the role of the factorization as the main conceptual and technical tool [29, 49–52].

While the boson–fermion equivalence appeals to the imagination, the existence of the supersymmetric structures in the real world has been discovered in much more modest circumstances. Thus, e.g., the non-relativistic Pauli electron in a homogeneous, time-independent magnetic field $\vec{B}$ pointing along the z-axis, with the Landau gauged vector potential $\vec{A} = (0, xB, 0)$, obeys the Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} - \frac{\mathrm{e}}{c}\vec{A}\right)^2 - \frac{\mathrm{e}}{2mc}\vec{B}\cdot\vec{\sigma} \tag{4.8}$$

which after elementary transformations is reduced to

$$H_\zeta \equiv \left(\frac{p_\zeta^2}{2m} + \frac{m}{2}\omega_B^2\zeta^2\right) - \frac{\omega_B}{2}\sigma_z \tag{4.9}$$

where $\omega_B = \frac{\mathrm{e}B}{mc}$ and $\zeta = x - \frac{c}{\mathrm{e}B}p_y$ (the inessential variables are separated and an additive constant renormalized). The oscillator and spin parts of (4.9) define Landau and Pauli levels. The particular value of the electron magnetic moment implies that the Landau and spin spacings are exactly the same, yielding a typical supersymmetric spectrum [49, 53, 54] (see also the complex equivalent [55]). Simple magnetic models for the general case of (4.7) exist as well for the Pauli electron in the external potential $V(x) = \alpha^2(x)$ associated with the inhomogeneous magnetic field of intensity $B(x) = 2\alpha'(x)$ orthogonal to the axis x (but then, the strict relation between the potential and the magnetic parts must be *a priori* assumed) [51, 53, 55, 56]. For the case of a magnetic monopole, see [57]; for the atomic and nuclear problems, e.g. [58, 59]. Note also the relevant scenarios based on the Dirac equation [49, 60–66]. The existence of the magnetic models encourages the conclusion that 'supersymmetry exists in nature' [49]. Yet, this is not exactly the same as to confirm the original idea about the boson–fermion equivalence [40, 41].

Indeed, some gaps in the analogy call attention. The elementary models [48, 49, 51, 53, 56] belong to orthodox quantum mechanics, where there are no fundamental doubts as to the physical reality of all states and observables. Thus, any self-adjoint operator in the Hilbert space $\mathcal{H}$ represents a legitimate observable, and any vector $|\psi\rangle \in \mathcal{H}$ is a physical state which can be produced by an adequate dynamical process. Even if it is not so easy to obtain an atomic electron in a superposed state of occupying simultaneously two different energy levels, such states are efficiently created by Rabi rotations in microwave cavities [67]. The 'inverted free evolution' $\mathrm{e}^{+\mathrm{i}\tau p^2/2}$, $\tau > 0$ and the squeezing operator $U_s = \mathrm{e}^{\mathrm{i}\lambda\frac{pq+qp}{2}}$, $s = \mathrm{e}^{-\lambda}$ do not resemble the standard evolution operations; yet both can be induced by time-dependent magnetic fields [68–71]. In general, in a well-equipped laboratory, the quantum mechanical systems (of positive energy) are completely manipulable [72–74] (in spite of the 'difficult' configurations [75, 76]), a fact of considerable interest for quantum control computing [77, 78].

An analogous structure for the genuine boson–fermion supersymmetry is far from obvious. Since there is no superselection rule, each degenerate subspace of H, apart from the Fock vectors, must contain its coherent superpositions. Thus, e.g., the first degenerate eigensubspace ($E = E_1$) spanned by $|1_B, 0_F\rangle$, $|0_B, 1_F\rangle$ contains all non-trivial combinations

$$\xi_0|1_B, 0_F\rangle + \xi_1|0_B, 1_F\rangle, \qquad |\xi_0|^2 + |\xi_1|^2 = 1, \qquad 0 \neq \xi_0, \xi_1 \in \mathbb{C}, \tag{4.10}$$

each one intuitively interpretable as a *hybrid particle*, a superposition of a *boson* and *fermion* (less provocative: an entangled state of the field superposed of having one boson and one fermion.) In fact, in each n-particle subspace $\mathcal{H}_n$, the 'hybrids' appear as the eigenvectors of the Hermitian supercharges

$$Q_\varphi = \mathrm{e}^{\mathrm{i}\varphi}Q_+ + \mathrm{e}^{-\mathrm{i}\varphi}Q_- \tag{4.11}$$

defining pairs of eigenstates $|\pm\rangle = \frac{1}{\sqrt{2}}(e^{i\varphi/2}|1,0\rangle \pm e^{-i\varphi/2}|0,1\rangle)$ with the eigenvalues $\pm\sqrt{E_n}$, each one representing a legitimate orthonormal basis ('albeit impure', says DeWitt [79], section 5.7, p 291). Can they be physically observed? The chances look slim. The spinorial charges are not measurable (Haag [80]). While still in the twice degenerate energy subspace of H, the 'boso–fermion', in a sense, is in an embryonic state, in which the characteristics of an adult individual are invisible. If perturbed, the specimen incubates, but then the supersymmetry is broken. The operational evidence of its previous nature is practically lost [46, 47]. A different chapter of the theory is the hypothesis about the need for super-Hilbert spaces [79], still without any experimental verification.

Yet, there is something incredible in the numerical coincidence between the Landau levels and the Bohr magneton, even if spoiled by radiative corrections. How many 'baby universes' should have been created to assure such an accidental coincidence? Whatever the mechanism, it means that the story must continue.

5. Quest for exact solutions

In fact, after recognizing its own status as the SUSY QM and adopting the Darboux heritage, the intertwining progressed fast. The transformed Hamiltonian models were carefully analysed in a sequence of 1985–87 papers of Sukumar [52, 81–84] without any *a priori* assumptions concerning the number of bound states. As a side effect, Sukumar's studies show a notable advantage of the new technique as compared to the previous inverse algorithms [22, 85]. In what follows, the concept of the SUSY QM visibly extends: it refers rather to the general Darboux intertwining than to a specific spectral structure. Indeed, it seems that each step of the factorization traduces itself into some spark of inspiration for other domains. The following incomplete list illustrates the phenomenon.

5.1. Higher order supersymmetry

Since Infeld and Hull [10–12] it has been obvious that the intertwinings (now called the Darboux transformations) can be iterated. Some premises about the higher order intertwining operators A, $A^\dagger$ already appeared in 1984 [24, 86]. In 1992, the second- and third-order differential operators A were applied by Dubov *et al* to obtain new Hamiltonians with equally spaced spectra, $HA = A(H+\omega)$ [87, 88]. They arrive at families of new potentials with 'ladder spectra', though without granting *a priori* that each ladder is only one and that it is infinite (see also [89, 90]). The subject was almost simultaneously approached by Veselov and Shabat [91], who presented a general theory of supersymmetric chains, including the case of equally spaced spectra.

The independent methods of 'comparative anatomy' permitting the evaluation of the shape of the Darboux transformed potentials without too heavy analytic machinery have been developed by Zakhariev *et al* [92–95]; they reveal the intricate forms and abundance of the new exact solutions in QM; see also their study of multichannel phenomena [96–98].

5.2. The polynomial algebra

The corresponding higher order algebra was designed in 1993 by Andrianov *et al* [99]. Suppose a Hamiltonian H_0 is intertwined with a chain of Hamiltonians $H_1, \ldots, H_n$ by a sequence of the first-order differential operators (2.6). Assume, however, that we are interested only in the initial and final Hamiltonians $H = H_0$ and $\widetilde{H} = H_n$ and we wish to interpret the transition

$H \to \widetilde{H}$ as a single intertwining step implemented by the nth-order differential operators A, $A^\dagger$, where $A = A_1 A_2 \cdots A_n$:

$$AH = \widetilde{H} A, \qquad A^\dagger \widetilde{H} = H A^\dagger. \tag{5.1}$$

Can we interpret H and $\widetilde{H}$ as a supersymmetric pair? Due to a (partial) isospectrality of H and $\widetilde{H}$, the answer is positive, but it means a generalization of the supersymmetric algebra (4.5), (4.6). Indeed, consider the new 'supercharges' and the Hamiltonian H:

$$Q_+ = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q_- = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}; \qquad \mathsf{H} = \begin{pmatrix} \widetilde{H} & 0 \\ 0 & H \end{pmatrix}. \tag{5.2}$$

Then $Q_\pm^2 = [Q_\pm, \mathsf{H}] = 0$, but

$$\{Q_+, Q_-\} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} \tag{5.3}$$

where none of $A^\dagger A$, $AA^\dagger$ coincides with H or $\widetilde{H}$. Yet, due to (5.1)

$$A^\dagger AH = A^\dagger \widetilde{H} A = HA^\dagger A, \tag{5.4}$$

and similarly $AA^\dagger \widetilde{H} = \widetilde{H} AA^\dagger$; i.e., $A^\dagger A$ and $AA^\dagger$ commute with H and $\widetilde{H}$, respectively. An inductive argument [91, 100] shows that $A^\dagger A$ and $AA^\dagger$ must be nth-order polynomials of H and $\widetilde{H}$: $A^\dagger A = f(H)$, $AA^\dagger = g(\widetilde{H})$. Moreover, (5.1) implies $f(H) = A(A^\dagger A) = (AA^\dagger)A = g(\widetilde{H})A = Ag(H) \Rightarrow A^\dagger A f(H) = A^\dagger A g(H) \Rightarrow f^2(H) = f(H)g(H)$; so for any eigenvalue E of H (discrete or continuous) one must have $f^2(E) = f(E)g(E)$. Since the finite-order differential operators in $L^2(\Omega)$ have infinite sets of spectral values, both polynomials coincide $f \equiv g$. Hence, one ends up with a generalized (*polynomial*) SUSY algebra (5.1)–(5.3), where (5.3) reads

$$\{Q_-, Q_+\} = \begin{pmatrix} f(\widetilde{H}) & 0 \\ 0 & f(H) \end{pmatrix} = f(\mathsf{H}). \tag{5.5}$$

If, moreover, the intertwining operator A is a product of n first-order steps, $A = A_n, \ldots, A_1$, intertwining the Schrödinger's Hamiltonians H_j, H_{j+1} *via* the factorization constants λ_j, $j = 1, 2, \ldots$ (2.7), then $f(\mathsf{H}) = (\mathsf{H} - \lambda_1), \ldots, (\mathsf{H} - \lambda_n)$. If the chain of intertwinings closes $AH = HA$, then Hamiltonian H possesses a non-trivial internal symmetry, interpretable in terms of the integrable Hamiltonian systems [101]. For n odd, it distinguishes a class of exceptional potentials with a final number of spectral gaps [91, 102] (one of the surprising ways of defining the special functions!). If the chain produces $\widetilde{H} = H + w \Leftrightarrow A(H + w) = HA$, the required symmetry of H leads to the Painlevé potentials (see Veselov and Shabat [91] and Adler [103]).

While some hints about the polynomial structure (5.5) appear earlier [87], its mature formulation was given in [99] and completed in [104] (see also [105, 106]).

5.3. *The squeezed intertwining and further progress*

In a different type of intertwining A, $A^\dagger$ are no longer finite-order differential operators, but involve a squeezing $A = \frac{1}{\sqrt{2}}(ip + \alpha(x))U_s$, with $U_s^\dagger x U_s = sx$, $U_s^\dagger p U_s = s^{-1}p$, the SUSY partner of H_0 is not a new Schrödinger's Hamiltonian, though it is proportional to one:

$$s^2 A H_0 = H_1 A. \tag{5.6}$$

The phenomenon was first studied by Spiridonov [107] (see also [108, 109]); it seems the unique case when the spectrum of H_0 is proportionally deformed (i.e. the spectral values $E_n^{(1)} = s^2 E_n^{(0)}$, $n = 1, 2, \ldots$). If moreover H_1 coincides with $H_0 + \omega$, then (5.6) tells that the spectrum of H_0 is conformal; if E is an eigenvalue of H_0 with an eigenvector $|\phi\rangle$ and $A|\phi\rangle$ has a finite, non-vanishing norm, then $s^2(E+\omega)$ is a spectral value as well; in particular, if $\omega = 0$, the spectrum has the tendency to form geometric sequences. A tempting question is, whether some more general functions of H could be obtained by intertwining

$$AH = \phi(H)A. \tag{5.7}$$

Such spectral transformations, though, would require different types of intertwining, since evidently (5.7) becomes impossible if ϕ is a polynomial and A is a finite-order differential operator (the orders of derivatives on both sides disagree!)

More general algebras too deserve attention. Following the SUSY formalism a natural idea was to use the same language to describe the (hypothetical) parastatistical phenomena. Here, the role of the fermionic creation–annihilation operators $\sigma_\pm, \sigma_+^2 = \sigma_-^2 = 0$ is assumed by their analogues with the longer n_F-ladders, i.e. nilpotent $a_\pm$ with $a_\pm^{n+1} = 0, a_\pm^n \neq 0$ (no more than *n parafermions* permitted in one state). The idea was proposed by Rubakov and Spiridonov [110], who designed the supercharges $Q_\pm$ in the form of nilpotent matrices with bosonic entries. Alternative models were soon proposed by Beckers and Debergh (an exact parasupersymmetry, triple degeneracy [111, 112], a parasuperspace [113]) and by Durand and Vinet (cyclotronic and Morse models [114], the spin 1 representation of $a_\pm$ [115]); another interesting approach is due to Plyushchay [116, 117]. It is also worth noting that the fundamental subject of *anyons* (see Goldin *et al* and Wilczek [118–120]) admits as well a natural algebraic treatment [121, 122].

5.4. The problem of reducibility

The question as to whether the *n*th-order intertwining can always be reduced to the conventional first-order steps was raised in 1995 by Andrianov *et al* [104] and was systematically examined by Bagrov and Samsonov [100]. As found in [100] each finite-order intertwining operator A splits naturally into a chain product of *irreducible* first- and second-order steps. Each first-order step is of traditional form $A_\mathrm{i} = \frac{1}{\sqrt{2}}(\mathrm{i}p + \alpha_i)$, intertwining two subsequent (conventional) Schrödinger's Hamiltonians H_i, H_{i+1}. Each irreducible second-order term $A_j = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left\{a(x), \frac{\mathrm{d}}{\mathrm{d}x}\right\} + b(x)$ can be decomposed into a product of two first-order intertwiners, but the price is that they expand the SUSY chain, inserting between H_j and H_{j+1} a new atypical Hamiltonian h_j ($H_j \rightarrow h_j \rightarrow H_{j+1}$) which is either complex or contains a new singularity (while H_{j+1} is again orthodox!). As found in [100] the complex h_j typically appear for pairs of complex roots of f in (5.5), a phenomenon which has lately awoken increasing interest (see section 7).

5.5. The double SUSY step

The reduction of the SUSY chains to the elementary first- and second-order steps turned more attention to the 'paso doble' of the supersymmetry, i.e. the second-order Darboux–Crum transformation [19]

$$V(x) \rightarrow \widetilde{V}(x) = V(x) - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln W(u_1, u_2) \tag{5.8}$$

where $W = u_1 u_2' - u_2 u_1'$ is the Wronskian of u_1, u_2. In the traditional first-order steps a lot of care was taken to apply the intertwining without introducing an extra singularity [123]. To achieve this, the Darboux transformations (3.1) were typically generated by (unphysical) eigenfunctions u of the initial Hamiltonian H_0, with the 'eigenvalues' ε below the ground state energy E_0 (of course, if $\sigma(H_0)$ is bounded from below). If $\mathbb{R} \ni \varepsilon > E_0$ (e.g. ε between two energy levels E_0, E_1) the transformation could not be carried out without introducing a new singularity, requiring a redefinition of the domain and of the Hilbert space itself. Yet, as detected in Krein [124], Sukhatme [125], Samsonov [126], Fernández [127] and proved generally by Samsonov [128], this limitation does not concern the second-order Darboux steps (5.8). Indeed, let H_0 be a Schrödinger Hamiltonian and $u, \widetilde{u}$ two nontrivial (not necessarily normalizable) real solutions of the eigenvalue equations

$$H_0 u = \varepsilon u; \qquad H_0 \widetilde{u} = \widetilde{\varepsilon}\widetilde{u}, \qquad \varepsilon < \widetilde{\varepsilon}. \tag{5.9}$$

The regularity of the second-order Darboux transformation induced by u and $\widetilde{u}$ depends on the absence of the nodal points of the Wronskian $W(x) = u\widetilde{u}' - \widetilde{u}u'$, where (5.9) assures that $W'(x) = (\varepsilon - \widetilde{\varepsilon})u\widetilde{u}$. A key step is now to choose $u(x)$ with $n + 1$ nodal points $\nu_1, \ldots, \nu_{n+1}$, separated by n nodal points $\widetilde{\nu}_1, \ldots, \widetilde{\nu}_n$ of $\widetilde{u}$:

$$\nu_1 < \widetilde{\nu}_1 < \nu_2 < \cdots < \widetilde{\nu}_n < \nu_{n+1}. \tag{5.10}$$

The configuration (5.10) looks atypical (were $u(x), \widetilde{u}(x)$ normalized eigenvectors with $\varepsilon < \widetilde{\varepsilon}$, then $\widetilde{u}$ would have more roots than u). However, $u, \widetilde{u}$ belong to wider 'unphysical eigenspaces', where (5.10) can occur for $\varepsilon, \widetilde{\varepsilon}$ in the same resolvent interval (E_k, E_{k+1}), $E_k < E_{k+1}$ [129]. Adopting the array (5.10), Samsonov shows that $W'(x)$ changes the sign $2n + 1$ times as x crosses the nodes (5.10), so it has opposite signs in $(-\infty, \nu_1)$ and in $(\nu_{n+1}, +\infty)$. Moreover, the signs of $W'(x)$ for $x > \nu_{n+1}$ and $x < \nu_1$ coincide and anticoincide, respectively, with the signs of $W(\nu_{n+1})$ and $W(\nu_1)$, hence $|W(x)|$ is an increasing function of $|x|$ for $x < \nu_1$ and $x > \nu_{n+1}$. It means that $W(x)$ cannot have nodes in $(-\infty, \nu_1] \cup [\nu_{n+1}, +\infty)$. Further arguments show that W has no roots in $[\nu_1, \nu_{n+1}]$ (see [128]). So, W has no roots in $\mathbb{R}$ and must generate a nonsingular second-order Darboux transformation. It is interesting that, for the opposite configuration, i.e., if u has n roots and $\widetilde{u}$ has $n+1$, the last statement might also be true, but the proof requires a detailed study of the asymptotic behaviour of $V_0(x)$ (see [130]).

The subject was independently addressed in a sequence of studies of the second- and higher order Darboux transformations [131–140] (see also the ample research reported in [53, 141, 142] and the literature quoted there.) In particular, the confluent case $\widetilde{\varepsilon} \to \varepsilon$ [143] seems of interest: if two encrusted levels coincide, the double Darboux step permits the construction of a class of the exceptional oscillating potentials which vanish for $x \to \pm\infty$, but admit bound states sustained by multiple reflection from $V(x)$ minima and maxima as $x \to \pm\infty$; see [125, 144, 145].

5.6. Periodic potentials

The Darboux methods are not limited to the discrete spectra, they can also be used to 'sculpt' the periodic potentials. By applying the single Darboux step (4.7) with $\varepsilon \leqslant E_0$, where E_0 is the 'ground energy' (i.e. the lower bound of the spectral continuum) one arrives at a new nonsingular $V_1(x)$, in general aperiodic and non-locally deformed (cases when periodicity is not lost can be as interesting [146]). By applying a double 1-SUSY step (5.8) with a pair of the factorization constants $\varepsilon_1 < \varepsilon_2$ in one of the spectral gaps $[E_n', E_{n+1}]$, one can also insert into the gap two discrete energy levels representing the pair of bound states created by the lattice impurity [147, 148]. As interesting are the techniques of inserting discrete levels inside the spectral bands [125].

An intriguing effect can show up if one applies chains of many first-order Darboux transformations (the 'dressing chains', cf Veselov and Shabat [91]). For some potentials the chain can close ($H_n = H_0$); a curious phenomenon which distinguishes special functions with finite numbers of spectral gaps. The complete theory exceeds this report (but see Veselov, Shabat [91] and Khare and Sukhatme in this issue). Let us only mention a simple case,

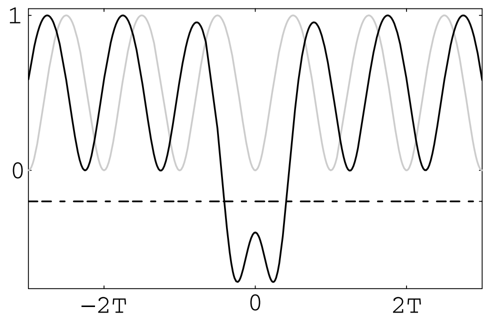

Figure 1. The Lamé potential $V(x) = 2m\mathrm{sn}^2(x|m)$ with $m = 0.5$ (grey) and its first-order regular deformation (black) with a bound state energy level (contact effect) at $\varepsilon = -0.2 < E_0 = 0.5$ (dashed).

concerning an apparently frustrated effect of some Darboux transformations [149–152]. The effect is that for some potentials, the Darboux transformation has almost no effect!

Indeed, let $H_0 = \frac{p^2}{2} + V(x)$, where $V(x)$ is periodic and at least twice differentiable. Suppose we apply $A = \frac{1}{\sqrt{2}}\big(\frac{\mathrm{d}}{\mathrm{d}x} + \alpha(x)\big)$, getting $AH = \widetilde{H}A$, where $\widetilde{H} = \frac{p^2}{2} + \widetilde{V}(x)$. The computer simulations show that for a class of periodic potentials $V(x)$, the Darboux transformed $\widetilde{V}(x)$ is just rigidly displaced: $\widetilde{V}(x) = V(x+\delta)$, where δ can take continuous values depending on the choice of $\alpha(x)$. The result looks disappointing: typically, one uses the Darboux method to transform rather than preserve the form of $V(x)$. Yet, why precisely can it happen? By writing the Riccati equations interrelating $\alpha(x)$ with $V(x)$ and $\widetilde{V}(x) = V(x+\delta)$ one easily shows [153]

$$\alpha^2(x) = V(x) + V(x+\delta) - 2\varepsilon \tag{5.11}$$

$$\alpha'(x) = V(x+\delta) - V(x). \tag{5.12}$$

Since different ε correspond to different δ, denote $\varepsilon = \varepsilon(\delta) \equiv -2\xi(\delta)$. Assume now that $V(x)$ is even and choose the new variables $u = x$, $v = -\delta - x$; simple calculation (differentiate (5.11) and compare with (5.12)) yields

$$\xi(u+v) + V(u) + V(v) = \frac{1}{4}\left[\frac{V'(u) - V'(v)}{V(u) - V(v)}\right]^2. \tag{5.13}$$

As one can observe (5.13) coincides with the well-known addition law for the elliptic functions, if $\xi(\delta)$ coincides with the Weierstrass function $\wp(\delta)$ (which is well assured by (5.11)–(5.13), see [153]). It looks as if the addition laws (5.13) for the elliptic functions were waiting about 100 years to reveal their supersymmetric sense. Can this be of some use? As it seems, it can. The eigenfunctions of H which generate the Darboux displacements (5.11), (5.12) of $V(x)$ are the nontrivial Bloch solutions $u_\pm(x)$ of $Hu = \varepsilon u$ in the (unphysical) resolvent set of H with the 'factorization energy' $\varepsilon < E_0$. Of them u_+ tends to ∞ as $x \to +\infty$, generating a positive displacement, while the other $u_- \to \infty$ for $\to -\infty$, generating a negative δ. The remaining (unphysical) eigenfunctions $u = c_+u_+ + c_-u_-$ ($c_\pm \in \mathbb{C}$) have no nodal points in $\mathbb{R}$ but diverge for both $\to \pm\infty$. By choosing one such untrivial u with $c_\pm \neq 0$ as a Darboux generator, one ends up with a nonlocal deformation shifting $V(x)$ to the right as $x \to -\infty$ and to the left as $x \to +\infty$ (or vice versa); thus producing a non-periodic $\widetilde{V}(x)$ in which two contradictory displacements either collide or diverge (see figure 1), giving the *contact effect* or a *quantum well* in the middle [154]. The new results for the second-order SUSY displacements were recently obtained by Samsonov *et al* [155].

5.7. The finite differences

The finite difference analogues of the differential operators are as interesting and intrigued many authors, including Euler [156]. The three-term recurrences of QM are indeed the second-order difference equations; albeit the correspondence between the difference and differential domains is not one-to-one. Thus, e.g., in the linear space of functions $\psi : \mathbb{R} \to \mathbb{C}$, with the multiplication by x and displacement operators D defined as $(x\psi)(x) = x\psi(x)$ and $(D\psi)(x) = \psi(x+1)$, the operations xD^{-1} and D commute as $[D, xD^{-1}] = \left[\frac{\mathrm{d}}{\mathrm{d}x}, x\right] = 1$ (if $x \in \mathbb{Z}$, the analogy with $A, A^\dagger$ is imminent; the representations in terms of higher order differential operators are as natural [157–160]). Finer imitations of x and $\frac{\mathrm{d}}{\mathrm{d}x}$ are also available. Taking $\nabla_h = \frac{D_h - 1}{h}$, where $(D_h f)(x) = f(x+h)$, one has

$$\left[\nabla_h, xD_h^{-1}\right] = 1; \qquad \nabla_h \underset{h\to 0}{\longrightarrow} \frac{\mathrm{d}}{\mathrm{d}x}, \qquad xD_h^{-1} \underset{h\to 0}{\longrightarrow} x \tag{5.14}$$

which intervenes in finite difference analogues of the traditional quantum mechanical problems (see, e.g., Toda [161], Dubrovin *et al* [26, 27, 102], Turbiner, Chryssomalakos [162, 163], Suzko *et al* [164, 165], Reyes and Rosu [166], Beals *et al* [167, 168]). Simultaneously, the finite difference techniques focused attention on q-deformed structures, whose central element is the ∂_q-derivative

$$\partial_q \psi(x) = \frac{\psi(x) - \psi(qx)}{(1-q)x}, \qquad 0 < q < 1 \tag{5.15}$$

(see, e.g., Odzijewicz [169]; a more symmetric definition, cf [170]). The ∂_q has some notable past. An intuitive integration inverting (5.15) was *de facto* performed by Archimedes to find the surface bordered by a parabola; Fermat used the geometric partition $a, qa, q^2a, \ldots,$ of the interval $[0, a]$ to integrate $f(x) = x^\kappa$; the formal definitions were given subsequently (cf [171]).

The intertwining of finite difference operators facilitates remarkably the solution of recurrence problems [172]. The generalization of q-deformed systems [107, 108] permits the construction of chains of finite difference Hamiltonians H_k with $A_k H_k = q_k H_{k-1} A_k$, of considerable interest to quantum optics [161, 169, 170, 172, 173].

5.8. The debate on coherent states

The spectral structure is not the only subject in SUSY QM. In all intertwined systems the supersymmetry brings valuable data about the time evolution of quantum states. For the traditional harmonic oscillator a notable phenomenon is the existence of specially regular Gaussian states $|\xi\rangle$, for which the Heisenberg uncertainty achieves its lower bound $\Delta x \Delta p = \hbar/2$, and the packet centres draw the classical phase trajectories. While the advantages of Gaussian packets were known for a long time [174], it was not noted till 1963 [175] (see also [176]) that they are the eigenstates of the (non-Hermitian) operator $A = \frac{1}{\sqrt{2}}(\mathrm{i}p + x)$:

$$A|\xi\rangle = \xi|\xi\rangle, \qquad \xi \in \mathbb{C} \tag{5.16}$$

and moreover, $[A, A^\dagger] = 1$ implies the analytic expression

$$|\xi\rangle = \mathrm{e}^{\xi A^\dagger}|0\rangle. \tag{5.17}$$

The states $|\xi\rangle$ are nonorthogonal and overcomplete [177]. As found subsequently (Klauder [178], Bargman [179] and Perelomov [180]) they are an example of certain general group theoretical design. Given a Lie group G (e.g. of Heisenberg–Weyl) with a left-invariant measure μ and an irreducible unitary representation $G \ni g \to U(g) = U(g^{-1})^\dagger$, and given a fixed state $|\theta\rangle$, the group driven states $|g\rangle = U(g)|\theta\rangle$ form an overcomplete system providing a decomposition of unity

$$I = \int_{\bar{G}} |g\rangle\langle g| \, \mathrm{d}\bar{\mu} \tag{5.18}$$

where $\bar{\mu}$ is induced by μ on the group quotient $\bar{G} = G/\Theta$ defined by the isotropy subgroup Θ of $|\theta\rangle$ (cf [180], section 2.3). Note though, that (5.18) holds generically, no matter the choice of $|\theta\rangle$, and tells little about the Heisenberg uncertainty $\Delta x \Delta p$ for the 'coherent states' $|g\rangle$; so (5.18) is not conclusive to identify the traditional coherent states of the harmonic oscillator. This leads to some open problems if one tries to construct the 'coherent analogues' of (5.16), (5.17) for an anharmonic $\widetilde{H}$. Indeed, $\widetilde{H}$ may admit various families of 'nice states', but none joining all virtues, i.e.: (a) minimal uncertainty, (b) satisfying (5.16), (c) forming an overcomplete basis in $\mathcal{H}$. This has opened the way for various competing definitions, following different philosophies.

In 1994–1999 Fernández *et al* adopted the idea that the best 'coherent states' for the Abraham–Moses (AM) anharmonic oscillators should be constructed by employing the ladder operators natural to this family. The AM potentials are strictly isospectral to (2.1) but the natural 'creation' and 'annihilation' operators $A^\dagger, A$ are of the third order [24]. They intercommunicate only the excited states of $\widetilde{H}$, leaving its ground state isolated. By postulating (5.16) Fernández *et al* [157] obtain a family of non-Gaussian packets, forming an overcomplete basis in the subspace of the excited states $|\theta_1\rangle, |\theta_2\rangle, \ldots,$ ($|\theta_0\rangle$ excluded), with the Heisenberg uncertainty slightly above $\hbar/2$. A compact formula analogous to (5.17) is obtained by defining the operator $B^\dagger = b^\dagger a^\dagger (N+1)^{-1}(N+2)^{-1} b$ which commutes with A to $[A, B^\dagger] = 1$ on the subspace $\mathcal{H}_1 = |\theta_1\rangle \oplus |\theta_2\rangle \oplus \cdots$, and leading to $|\xi\rangle = \mathrm{e}^{\xi B^\dagger}|0\rangle$. In [158–160] the construction is improved by defining $C_w = b^\dagger f(N) ab$, with $\left[C_w, C_w^\dagger\right] = 1$ in $\mathcal{H}_2$, with an arbitrary parameter w allowed in the construction of C_w.

The idea has been questioned by Kumar and Khare (K–K) [181] who propose a different construction of the 'true' coherent states, based on the isospectrality of the traditional and *distorted* oscillators H and $\widetilde{H}$ in [24]. Since $\sigma(H) = \sigma(\widetilde{H})$, there exists a unitary transformation U such that $\widetilde{H} = U^\dagger H U$; K–K believe it is most natural to apply U to the coherent states of H constructing the coherent states of $\widetilde{H}$, and therefore, they consider the states defined in [157–159] 'incorrect'.

The discussion illustrates indeed the fact that for the transformed systems the concepts 'split'. Each 'distorted potential', in general, admits several classes of nicely behaving states. It is somewhat platonic to *prove* what *must be* the coherent states, if one has no generic definition. In fact, this is precisely the centre of the problem! Since one does not prove definitions, the best philosophy, perhaps, is that of 'all flowers'. Thus, e.g., the K–K states for the AM potentials enjoy some pleasant properties: they form an overcomplete basis in the entire $\mathcal{H} = L^2(\mathbb{R})$ and are associated with the ladder operators which do not omit the new ground state $|\theta_0\rangle$. However, the idea can be implemented only in special cases when the initial and transformed Hamiltonians H and $\widetilde{H}$ are exactly isospectral. If some new (arbitrary) energy levels $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are added, then the images of the coherent states of H do not span the entire space, as they cannot change their original evolution frequencies.

In contrast, the approach by Fernández *et al* starts 'from the little', respecting the SUSY structure and following the heuristic steps which have lead to the coherent fields in quantum optics [182]. As an unexpected reward their family has some new qualities (figure 2); in the limit $\widetilde{H} \to H$ it does not reduce itself to the orthodox coherent family, but yields new *meta*-coherent states of the old oscillator (we do not use the name 'super' which seems too

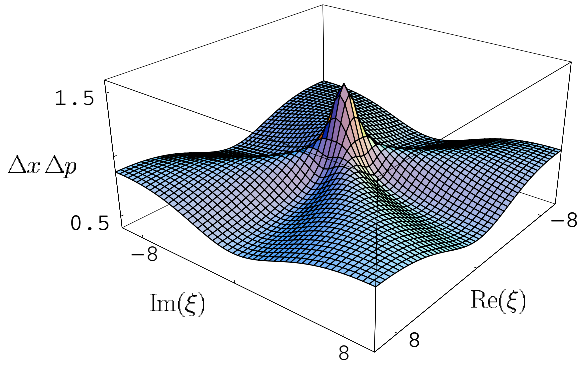

Figure 2. The uncertainty product for the disputed *meta*-coherent states $|\xi\rangle$ ($\xi \in \mathbb{C}$) of Fernández *et al*.

abused). As it seems, a similar approach works also for the coherent states of the transparent wells [183].

An independent quest for coherent states of arbitrary Hamiltonians is presented in an ample study of Spiridonov [184]. One of the ideas is that for any non-degenerate discrete spectrum Hamiltonian H, the coherent family $\xi \to |\xi\rangle$ should form a generating function of the sequence of eigenstates $|\theta_n\rangle$ (cf also Man'ko *et al* [185], Bagrov–Samsonov [183, 186], Seshadri *et al* [187], Penson and Solomon [188], Antoine *et al* [189]). As it seems, before making a final choice (if any) it is worthwhile looking at other levels of physical theory.

6. Is the whole truth in the channel of shallow water?

One of exceptional structures of SUSY QM is the *transparent wells* of Pöschl–Teller, the results of multiple application of the intertwining to the null potential $V_0 \equiv 0$. The so generated wells can host finite sets of bound states, but some qualities of the null potential remain; e.g., being perfectly visible for the trapped packets, the wells are completely invisible for the $|\text{in}\rangle$ states arriving from $\pm\infty$; the fact demonstrated easily by their vanishing reflection coefficients [138, 141].

The most intriguing aspect, however, is their double role: being the simplest solvable potentials in QM, the invisible wells serve simultaneously as the instantaneous $\tau = \text{const}$ profiles for the solutions of the well-known KdV equation

$$u_\tau + 6uu_x - u_{xxx} = 0 \tag{6.1}$$

which describes the evolution of the localized 'solitary waves' in the *channel of shallow water* [190], the fact which earns them the name of *soliton potentials* (note that, in the physical interpretation of the KdV waves, τ means the time but in the associated Schrödinger eigenproblem it is just a parameter). As found by Miura, the τ-evolution of the KdV waves $u(x,\tau)$ leaves invariant the spectrum of the corresponding Schrödinger operator

$$H_\tau = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x,\tau) \tag{6.2}$$

and the reason is that if u fulfils (6.1) then the τ-dependence of H_τ (via $u(x,\tau)$) traduces itself into a generic isospectral evolution $\frac{\mathrm{d}}{\mathrm{d}\tau}H_\tau = [A_\tau, H_\tau]$, induced by the auxiliary anti-Hermitian operator $A_\tau = -4\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 3\left(u\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}u\right) + 3u_x$. Thus, KdV turns out the consistency condition for the simultaneous validity of (6.2) and the isospectral drift of A_τ. Analogous observations allow us to solve other nonlinear equations such as sine-Gordon and cubic Schrödinger (see Lax [191], Levi [192], Rauch–Wojciechowski [193], the monographs of Lamb Jr [194], Matveev and Salle [195]). Simultaneously, as noted by Miura *et al* [196], Wahlquist and Estabrook [197], (6.2) admits Bäcklund transformations in the form of a nonlinear superposition law which permits one to determine new solutions of (6.1) in terms of the known ones. Quite notably, the Bäcklund techniques at the level of KdV mirror the quantum mechanical intertwining. To see this, forget for a moment about the KdV aspect of u and consider the Schrödinger Hamiltonian $H_0 = -\frac{d^2}{dx^2} + V_0$ where V_0 is an arbitrary potential intertwined with a family (infinite or finite) of new Hamiltonians $H_1(\varepsilon) = -\frac{d^2}{dx^2} + V_1(x, \varepsilon)$ by the first-order operators $A_1(\varepsilon) = \frac{d}{dx} + \alpha_1(x, \varepsilon)$, where ε are the factorization constants, i.e.: $H_0 = A_1^\dagger A_1 + \varepsilon$ and $H_1 = A_1 A_1^\dagger + \varepsilon$. Consistently, α_1 must fulfil the Riccati equations

$$-\alpha_1'(x, \varepsilon) + \alpha_1^2(x, \varepsilon) = V_0 - \varepsilon. \tag{6.3}$$

$$\alpha_1'(x, \varepsilon) + \alpha_1^2(x, \varepsilon) = V_1(x, \varepsilon) - \varepsilon. \tag{6.4}$$

Equation (6.3) means that $\psi_0(x, \varepsilon) = e^{-\int \alpha_1(x,\varepsilon)\, dx}$ satisfies (for all ε) the original eigenequation $H_0\psi_0(x, \varepsilon) = \varepsilon\psi_0(x, \varepsilon)$ (even though $\psi_0(x, \varepsilon)$ in general does not need to have a finite norm). Assume that the first-order intertwining (6.3), (6.4) is known for at least two different values of ε. Then, fixing ε and applying to $\psi_0(x, \varepsilon)$ the $A_1(\mu)$ for $\mu \neq \varepsilon$, we must obtain an eigenfunction of $H_1(\mu)$:

$$\begin{aligned} \psi_1(x, \epsilon, \mu) &= \left[\frac{d}{dx} + \alpha_1(x, \mu)\right]\psi_0(x, \varepsilon) \\ &= [-\alpha_1(x, \varepsilon) + \alpha_1(x, \mu)]\, e^{-\int \alpha_1(x,\varepsilon)\, dx}. \end{aligned} \tag{6.5}$$

Consistently, the function $\alpha_2(x, \varepsilon, \mu)$, defined by

$$\psi_1(x, \epsilon, \mu) = e^{-\int \alpha_2(x,\varepsilon,\mu)\, dx} \tag{6.6}$$

must satisfy the next Riccati equation

$$-\alpha_2'(x, \varepsilon, \mu) + \alpha_2^2(x, \varepsilon, \mu) = V_1(x, \varepsilon) - \mu. \tag{6.7}$$

Reading back the definition (6.5) and using (6.6) one has

$$\alpha_2(x, \varepsilon, \mu) = -\frac{d}{dx} \ln \psi_1(x, \varepsilon, \mu) = \alpha_1(x, \varepsilon) - \frac{\alpha_1'(x, \varepsilon) - \alpha_1'(x, \mu)}{\alpha_1(x, \varepsilon) - \alpha_1(x, \mu)} \tag{6.8}$$

and using again (6.3), one obtains the finite difference Bäcklund equation

$$\alpha_2(x, \varepsilon, \mu) = -\alpha_1(x, \mu) - \frac{\varepsilon - \mu}{\alpha_1(x, \varepsilon) - \alpha_1(x, \mu)} \tag{6.9}$$

which determines algebraically the superpotential for each next intertwining step under the condition that one knows at least two (alternative) previous steps with two different factorization constants. This seems the reason why many structuralists consider the intertwining, Darboux and Bäcklund transformations as practical synonyms, at least for all cases involving the Riccati equations (see [102, 103, 198, 199]).

In spite of these results some questions are open. It calls attention that the Bäcklund transformation (6.9) initially used to generate the 'multisolitonic wells', is not at all limited to the transparent potentials. It can be applied to any potential V generating the deformed versions $V+\delta V$. These deformations show some familiar patterns: the perturbations δV typically vanish at infinity (except the periodic or quasi-periodic cases), the number of minima and maxima increases as the transformation is repeated forming a qualitative soliton pattern (enough to compare with [141, 200, 201]). A question arises whether the SUSY deformations of an arbitrary potential are not the 'second class members' of the soliton community. In particular (i) is it possible to find for the SUSY deformed versions of any $V(x)$ some (presumably nonlinear) propagation equation, which would keep the τ-dependent Hamiltonians isospectral? (ii) If so, would some particular shapes of δV survive mutual collisions, as the solitons do on the background of $V_0 \equiv 0$? (iii) Would the SUSY deformations δV of any V_0 enjoy some transparency properties for a certain class of wave packets?

As to (i), (ii), there is no sign that the answer might be positive. Yet, (iii) could make sense if the concept were properly understood. Of course, it would be irrelevant to speak about reflection and transition coefficients for a packet circulating in a (transformed) oscillator, but on the other hand, the SUSY deformations are (almost) isospectral (see [90]) so they are invisible if one observes the higher spectral lines. This suggests some alternative transparency idea. As an example consider a 'solitonic well' $\widetilde{V}(x)$ with $\widetilde{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \widetilde{V}(x)$ coupled by an intertwiner A (of any order) with the free evolution $H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, i.e.,

$$A^\dagger H_0 = \widetilde{H} A^\dagger \Leftrightarrow H_0 A = A\widetilde{H}. \tag{6.10}$$

Then, one has also

$$A^\dagger \,\mathrm{e}^{\mathrm{i}\lambda H_0} = \mathrm{e}^{\mathrm{i}\lambda \widetilde{H}} A^\dagger, \qquad \mathrm{e}^{\mathrm{i}\lambda H_0} A = A\,\mathrm{e}^{\mathrm{i}\lambda \widetilde{H}}. \tag{6.11}$$

For the free Hamiltonian H_0 the motion of the t-dependent Heisenberg observables p and q reads

$$p(t) = \mathrm{e}^{\mathrm{i}tH_0} p\,\mathrm{e}^{-\mathrm{i}tH_0} = p = \text{const} \tag{6.12}$$

$$q(t) = \mathrm{e}^{\mathrm{i}tH_0} q\,\mathrm{e}^{-\mathrm{i}tH_0} = q + pt. \tag{6.13}$$

Following the idea of [157–160], consider the *meta*-observables

$$\widetilde{q} = A^\dagger q A, \qquad \widetilde{p} = A^\dagger p A \tag{6.14}$$

and ask, how do they evolve in the presence of the transformed Hamiltonian $\widetilde{H}$? Even though the mapping $H_0 \to \widetilde{H}$ is not unitary, one has

$$\begin{aligned}\widetilde{q}(t) &= \mathrm{e}^{\mathrm{i}t\widetilde{H}}\widetilde{q}\,\mathrm{e}^{-\mathrm{i}t\widetilde{H}} = \mathrm{e}^{\mathrm{i}t\widetilde{H}} A^\dagger q A\,\mathrm{e}^{-\mathrm{i}t\widetilde{H}} = A^\dagger\,\mathrm{e}^{\mathrm{i}tH_0} q\,\mathrm{e}^{-\mathrm{i}tH_0} A \\ &= A^\dagger (q + pt) A = \widetilde{q} + \widetilde{p}t\end{aligned} \tag{6.15}$$

and similarly,

$$\widetilde{p}(t) = \widetilde{p} = \text{const} \tag{6.16}$$

meaning that the evolution law for $\widetilde{q}(t)$, $\widetilde{p}(t)$ is not affected at all by the solitonic well $\widetilde{V}(x)$. Choosing an initial wave packet $|\psi\rangle$ which vanishes fast enough for $|x| \to +\infty$, one can assure that both average values $\langle\psi|\widetilde{q}|\psi\rangle$, $\langle\psi|\widetilde{p}|\psi\rangle$ are well defined. Now, if $|\psi\rangle$ evolves according to $\widetilde{H}$, its 'average velocity' stays constant and its 'renormalized centre' $\langle\psi|\widetilde{q}|\psi\rangle$ moves uniformly:

$$\langle\psi_t|\widetilde{q}|\psi_t\rangle = \langle\psi_0|\widetilde{q}|\psi_0\rangle + t\langle\psi_0|\widetilde{p}|\psi_0\rangle \tag{6.17}$$

so not only does the packet not suffer reflections but its 'meta-centre' $\widetilde{q}$ moves smoothly. This is no surprise if the packet represents the bound (stationary) state of the well with $\widetilde{p} = 0$, $\widetilde{q} =$ constant; a bit more unexpected if it has a part which is not trapped in the well (does it mean that the 'meta-centre' does not feel the obstacle?). An analogous phenomenon repeats itself if H_0 and $\widetilde{H}$ are the orthodox and deformed versions of the harmonic oscillator. Formula (6.14) is valid, but now $\widetilde{q}(t) = \widetilde{q}\cos wt + \widetilde{p}\sin wt$ and $\widetilde{p}(t) = -\widetilde{q}\sin wt + \widetilde{p}\cos wt$, the behaviour shared by the average values $\langle\psi_t|\widetilde{q}|\psi_t\rangle$, $\langle\psi_t|\widetilde{p}|\psi_t\rangle$. Does it mean that the SUSY deformation is in some sense invisible for the wave packet? (Cf Mentrup and Luban [202].)

While the questions remain open, the progress of SUSY QM continues in several other directions. The general Bäcklund idea, as outlined by Lamb [203], turns increasingly useful in relativistic theories. It has been applied, e.g., to transform the heavenly equations of the first into the second class [204, 205]. It is one of most promising techniques to solve the GR-equations in matrix form [206], it also turns out to be an efficient tool to clarify the structure of axially-symmetric solutions [207], to design the prolongation structures [208] and hyper-heavens [209]. The intertwining has been applied in cosmology to transform the normal modes [210]. Note also more implications in cosmology [211–215] as well as in nonlinear QM [216]. The applications of the Dirac equation [61, 62, 65, 66], to the matricial supersymmetry [217–221] and to the quantum Hamiltonians depending explicitly on time [100, 222–224] seem promising.

While these are the natural lines of expansion, the intertwining might also touch some more exotic problems.

7. Atypical models

To deform the traditional self-adjoint Hamiltonians is not the only option permitted by the method. In fact, the axioms about the Hermitian operators and real spectra can be abolished in some physical situations (cf [225, 226]). Thus, in orthodox QM the unstable states around the local potential minima can be reasonably described as the 'eigenfunctions' of the self-adjoint H with complex eigenvalues; the configurations called *Gamov vectors* [227, 228]. If $H = H^\dagger$, the evolution $\mathrm{e}^{-\mathrm{i}tH}$ conserves the norm, so the Gamov vectors cannot be normalizable (their norms could not be attenuated). They belong to 'rigged Hilbert spaces' [229, 230]. Note though, that they are not excluded from the factorization mechanism. The simplest example is the repulsive oscillator

$$H = \frac{p^2}{2} - \frac{x^2}{2} = AB - \frac{\mathrm{i}}{2} \tag{7.1}$$

where $A = \frac{1}{\sqrt{2}}(p+x)$, $B = \frac{1}{\sqrt{2}}(p-x)$, $B \neq A^\dagger$. Quite obviously, $HA = A(H - \mathrm{i})$, so starting from the 'vacuum' $B|\Gamma_0\rangle = 0 \Rightarrow |\Gamma_0\rangle = c_0\,\mathrm{e}^{\mathrm{i}\frac{x^2}{2}} \Rightarrow H|\Gamma_0\rangle = \gamma_0|\Gamma_0\rangle$, where $\gamma_0 = -\mathrm{i}/2$ and applying $A = -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\frac{x^2}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{e}^{-\mathrm{i}\frac{x^2}{2}}$ one generates a sequence of Gamov vectors $|\Gamma_n\rangle = c_n A^n|\Gamma_0\rangle = c_n h_n(x)\,\mathrm{e}^{\mathrm{i}\frac{x^2}{2}}$, with $h_n(x) = (-\mathrm{i})^n\,\mathrm{e}^{-\mathrm{i}x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\mathrm{e}^{\mathrm{i}x^2}$ and $\gamma_n = -\mathrm{i}\big(n + \frac{1}{2}\big)$. The $|\Gamma_n\rangle$ decay as $|\Gamma_n(t)\rangle = \mathrm{e}^{-(n+\frac{1}{2})t}|\Gamma_n\rangle$, meaning simply that the wavefunctions, repulsed from any finite region by $V(x) = -x^2/2$, escape to $\pm\infty$ (enough to check currents!). The example has no mysteries; it can be immediately derived by substituting $\rightarrow \sqrt{-\mathrm{i}}x$ in the ordinary attractive oscillator. Yet it shows that the Gamov vectors are not banished from the SUSY QM. Indeed, some 'repulsive Hamiltonians' can be easily constructed by replacing A and B in (7.1) by A and $\mathrm{i}B$ of Fernández *et al* [158], leading to the higher order differential operators with explicitly known Gamov states. Could the atypical AB factorization play a similar role for the Gamov spectra of the potential barriers as the traditional SUSY does for the potential wells?

In Gamov's theory the eigenvalues are complex but the Hamiltonians are real. This does not exhaust the physical reality, where the states can be unstable not only due to propagation or tunnelling, but also since the new reaction channels are open and the system 'migrates' from its initial Hilbert space. In this case, the effective Hamiltonian is no longer self-adjoint but can have normalized eigenvectors with complex eigenvalues $\lambda = E - \mathrm{i}\frac{\sigma}{2}$, with $\sigma > 0$ defining the escape rates (cf [231]). Apart from the radiative decay, the phenomenon admits quite elementary optical models. The simplest one is just the optical bench. Suppose, a beam

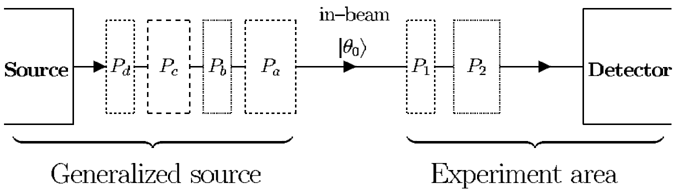

Figure 3. The absorbing filters in the way of particle beams can be represented by non-unitary linear operators [232].

of particles crosses a sequence of material obstacles (we imagine them as semitransparent particle filters.) Each obstacle absorbs a fraction of the beam and performs some operation on the rest. Assume that the operations, even if dissipative, do not mix states: if an incoming beam was pure, so will be the transmitted one. If no state is completely absorbed, then the action of each filter can be represented by a certain linear, non-unitary, non-singular operator $Y : \mathcal{H} \to \mathcal{H}$, which, in general, does not increase the norm (the norms mean the total beam intensity), i.e. $||Y\psi||^2 \leqslant ||\psi||^2$. A well-known example is the polarized photons penetrating through windows which transmit selectively distinct polarizations.

Since the beams partly perish, the operations performed by the filters cannot be considered unitary; yet each one is described by a certain linear state transformation representing the operation on the field vectors (see, e.g., [232]).

A certain tacit assumption is worth commenting on. When a bench operation is represented by a non-Hermitian Hamiltonian with eigenvalues $\lambda \in \mathbb{C}$, it is usually supposed that $\mathrm{Im}(\lambda) \leqslant 0$, meaning that the incident beam $|\theta_0\rangle$ can only lose particles. However, what is incident in one experiment might be the output of previous filtering operations (see figure 3). If we delete one of them, the intensity of the beam might well increase, so $\mathrm{Im}(\lambda) \geqslant 0$ should not be forbidden.

An intriguing ramification of the subject is the quest for the so-called *time operator* [233–238], motivated by the hope of arriving at a space–time quantization in which the space and time variables would have equal status [239, 240]. Until now, the idea has been frustrated by the fact that the ‘probability distribution on the time axis’ in the experiments with waiting detectors (e.g. screens) does not correspond to the spectral measure of any self-adjoint operator. Recent research points rather to the non-projective *POV-measures* [241–247], but even this intent has some disadvantages (see, e.g., the discussions in [247, 248]). As it seems, the only chance of describing the evolution of a micro-object in the presence of a waiting detector consists in introducing a non-Hermitian (normal) Hamiltonian with complex eigenvalues [232, 249–251].

The problem of complex eigenvalues has recently been approached by the intertwining methods [225, 226, 252–257]. Though merely taking its first steps, the idea might contribute, e.g., to the solvable optical models or perhaps, extend the multichannel studies of Zakhariev and Chabanov [92, 94, 97, 98] and perhaps approach the description of the ‘time operator’ [247]. The unsolved problem about the unitarity of the Cabibo matrix in QFT [258] is also worth remembering.

The complex eigenvalues are only a part of the non-Hermitian story. Over the past decades one can observe an increasing interest in complex potentials with real spectra, a phenomenon characteristic of PT-symmetric systems [259–264]. A similar effect can appear, under adequate asymptotic conditions, for real potentials perturbed by small imaginary parts, $V(x) + \mathrm{i}\phi(x)$ (see, e.g., Stepin [265]). An alternative source is also the atypical factorizations $H = AB + \epsilon$, where $A \neq B^\dagger$ [257, 266, 267]. All systems of this kind no longer respond to the Weisskopf–Wigner approach: though the eigenvalues are real, the eigensubspaces are not orthogonal and their physical interpretation presents an open challenge.

In the case of PT-symmetric Hamiltonians it has been noted [264, 268, 269] that they are pseudo-Hermitian, with the corresponding spectral implications [270, 271]. Yet, all intents to find a consistent probabilistic interpretation have been frustrated [272]. Still, further studies reveal links of the pseudo-Hermitian (pseudounitary) operators with multiple areas including cosmology (see Mostafazadeh [273]).

Recently Bender *et al* have formulated a new hypothesis concerning the diagonalizable, non-Hermitian operators $H \neq H^\dagger$ with real eigenvalues [274]. According to [274], the physical interpretation of such operators requires a basic re-definition of the Hilbert space metric, so that the eigensubspaces of H in a modified inner product become orthogonal. The idea must be taken under caution, as it might affect various levels of physical theory. On the most rudimentary level, the hypothesis seems to contradict the established paradigms. An advantage of the orthodox state-observable structure in a fixed Hilbert space geometry is that it can accommodate an infinity of self-adjoint QM observables, without the need to redefine the metric every time a new one is introduced. Yet, by going deeper, one faces unfinished structural discussions. In the quickly progressing modern theories (strings, quantum gravity etc) one can see a remarkable contrast between the flexibility of the cosmological–topological elements and rigidity of the quantum design, always repeating the same general scheme of self-adjoint observables and complex amplitudes (even if living on loops, branes, etc.) Looking from this perspective the effect predicted in [274] goes in a different direction; it seems analogous to the geometry deformation by the presence of matter in GR. In this case though, the structure affected would be the *quantum logic* [275–278], describing the elementary *yes–no measurements*. In the orthodox QM these measurements are represented by the orthogonal projectors in $\mathcal{H}$. The results 'yes' and 'no' correspond to the pairs of orthogonal subspaces, one of them rigidly determined by the other, which means that the negation of the logic is unique. If, however, the theory admits non-Hermitian diagonalizable operators with real spectra, the yes–no measurements could correspond to non-Hermitian projectors whose 'yes' and 'no' subspaces are not orthogonal, and the negation is no longer unique. The presence of diverse Hamiltonians (physical environment) could create many different ways of negating the same property (the time-dependent case, cf [279]).

In the orthodox axioms of *quantum logic* [275, 277] such flexible structures are not permitted. This is no longer so in the generalized descriptions based on the convex set geometry. Here, the fundamental object is the convex set S of all pure and mixed states of a certain quantum system [280, 281]. For a general S, the typical quantum concepts exist, but they may have a deformed structure. The 'questions' of the logic are now represented by *faces* of the convex set S: any given face, in general, possesses many complementary faces, illustrating many ways of applying the negation [282]. The entire structure has been described on a purely abstract level, without assuming any concrete theory; presumably, it could accommodate models where the Hilbert space metric is not absolute, but enforced by physical surroundings (see the polemic discussions in [232, 282–284]). All this, of course, must be taken with extreme caution. The future of the supersymmetric theories (including the super-manifolds [79]) is no more predictable than the consequences of the Darboux theorem and the Dirac stratagem were in 1882 and 1935. In fact, until now there are no Higgs bosons, no 'hybrid states', no modified spaces, etc.

Yet, it is an achievement of the factorization that it has turned our attention to a number of unsuspected structures. If one accepts the Wigner statement about the 'unreasonable efficiency of mathematics' [285], they have to materialize at some moment. If not, the question arises, what are they? Some abstract designs which are permitted to exist, but missed their chance? Or our glimpses of an unknown universe?

Acknowledgments

The authors are grateful to all colleagues from the Physics Department Cinvestav Mexico, for their critical and helpful discussions. Thanks are due to Maciej Dunajski, David Fernández, H García Compean, Juan J Godina, David Lowe, Anatol Odzijewicz, Maciej Przanowski, Mikhail Shubin and Jordi Sod Hoff. We would also like to honour all authors whose valuable works missed entering into our limited report. Thanks are due to the organizers of the International Conference PSQM'03, Valladolid, Spain, for the kind invitation and encouragement. The support of CONACyT (México), project 40888-F, is acknowledged.

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